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# A generalized $Q$ -operator for $U_q(\widehat{\mathfrak{sl}}_2)$ vertex models

Marco Rossi and Robert Weston

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK

E-mail: M.Rossi@ma.hw.ac.uk and R.A.Weston@ma.hw.ac.uk

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## Abstract

In this paper, we construct a  $Q$ -operator as a trace of a representation of the universal  $R$ -matrix of  $U_q(\widehat{\mathfrak{sl}}_2)$  over an infinite-dimensional auxiliary space. This auxiliary space is a four-parameter generalization of the  $q$ -oscillator representations used previously. We derive generalized  $T$ - $Q$  relations in which three of these parameters shift. After a suitable restriction of parameters, we give an explicit expression for the  $Q$ -operator of the 6-vertex model and show the connection with Baxter's expression for the central block of his corresponding operator.

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## 1. Introduction

Baxter's  $Q$ -operator has an interesting history. It was first constructed in 1972 as a tool for solving the 8-vertex model. The background was that the 6-vertex model had been solved by Bethe ansatz in the mid 1960s by Lieb and Sutherland [1–4]. However, this technique could not be simply extended to the 8-vertex model due to the absence of a suitable Bethe ansatz pseudo vacuum (a problem associated with the lack of 'charge conservation' through vertices for this model). Then, in a seminal series of papers [5–8], Baxter introduced his  $Q$ -operator<sup>1</sup> as an apparent *deus ex machina* which allowed him to write down Bethe equations for the eigenvalues of the 8-vertex model transfer matrix *without* having an ansatz for the eigenvectors (he did of course construct some eigenvectors using other techniques, see [6–8]).

His approach was to start with the 6-vertex model Bethe ansatz, and to derive certain functional relations between the transfer matrix  $T(v)$  and a matrix  $Q(v)$ —the elements of both matrices being entire functions. He went on to show that the reverse argument could be used in order to start from the functional relations (and some other properties of  $T(v)$  and  $Q(v)$ ) and derive the Bethe equations. He then considered the 8-vertex model, constructed a

<sup>1</sup> In fact, Baxter gives two different constructions of a  $Q$ -operator in [5] and [6–8]. We shall continue for the moment to use the generic term 'Baxter's  $Q$ -operator', and will specify to which construction we are referring when it becomes necessary to do so.

$Q(v)$  operator that obeyed the correct requirements, and used the reverse argument to derive Bethe equations. The approach is described clearly in Baxter's book [9].

Later on in the 1970s, the quantum inverse scattering method (QISM) was developed and used to produce a rather simpler derivation of the same Bethe equations for the 8-vertex model (the algebraic Bethe ansatz approach) [10]. Baxter also invented his corner transfer matrix technique for the 8-vertex model [9]. So, remarkably successful though it was, the  $Q$ -operator approach perhaps came to be considered by many as a historical curiosity.

However, in the last few years there has been something of a revival of interest in  $Q$ . The reasons for this include the following.

- Some understanding has been obtained into how  $Q$  fits into the QISM/quantum-groups picture of solvable lattice models [11–18].
- The discovery of the mysterious ODE/IM models correspondence—relating functional relations obeyed by the solutions and spectral determinants of certain ODEs to Bethe ansatz functional relations [19–21].
- The role of  $Q$  in classical integrable systems as a generator of Backlund transformation has been understood in certain cases (see [22] and references therein).

In this paper, we are concerned with the first point. The key to the QISM approach to solvable lattice models is to understand them in terms of an underlying algebra  $\mathcal{A}$ . The generators of  $\mathcal{A}$  are matrix elements  $\mathcal{L}^{ij}(z)$ , where  $i, j \in \{0, 1\}$  (in the simplest case) and  $z$  is a spectral parameter. The set of relations amongst the generators are given by the matrix relation

$$R(z/z')\mathcal{L}_1(z)\mathcal{L}_2(z') = \mathcal{L}_2(z')\mathcal{L}_1(z)R(z/z') \quad (1.1)$$

where  $\mathcal{L}_1(z) = \mathcal{L}(z) \otimes 1$ ,  $\mathcal{L}_2(z) = 1 \otimes \mathcal{L}(z)$  and  $R(z)$  is a  $4 \times 4$  matrix.

This QISM description was later refined in terms of quantum groups. In this picture  $\mathcal{A}$  is recognized as a quasi-triangular Hopf algebra (also known as a quantum group). For the vertex models of the title, the algebra  $\mathcal{A}$  is  $U_q(\widehat{\mathfrak{sl}}_2)$ . Families of  $R$ -matrices and  $\mathcal{L}$ -operators are then all given in terms of representations of a universal  $R$ -matrix  $\mathcal{R} \in U_q(b_+) \otimes U_q(b_-)$ , where  $U_q(b_{\pm})$  are two Borel subalgebras of  $U_q(\widehat{\mathfrak{sl}}_2)$ . The relevant  $U_q(\widehat{\mathfrak{sl}}_2)$  representations are the spin- $n/2$  evaluation representations  $(\pi_z^{(n)}, V_z^{(n)})$  defined in section 3 (in this paper, a representation of an algebra  $\mathcal{A}$  is specified by a pair  $(\pi, V)$ , consisting of an  $\mathcal{A}$  module  $V$  and the associated map  $\pi : \mathcal{A} \rightarrow \text{End}(V)$ ). Then we have

$$R(z/z') \equiv (\pi_z^{(1)} \otimes \pi_{z'}^{(1)})\mathcal{R} \quad \mathcal{L}(z) \equiv (\pi_z^{(1)} \otimes 1)\mathcal{R}$$

and (1.1) follows as a simple consequence of the Yang–Baxter relation for  $\mathcal{R}$ . More generally, we can define

$$\mathcal{T}^{(n)}(z) \equiv \text{Tr}_{V_z^{(n)}}(\mathcal{L}^{(n)}(z)) \quad \text{where} \quad \mathcal{L}^{(n)}(z) \equiv (\pi_z^{(n)} \otimes 1)\mathcal{R} \quad \text{and} \quad n \in \mathbb{Z}. \quad (1.2)$$

The  $\mathcal{T}^{(n)}(z)$  form a family of  $U_q(b_-)$  valued transfer matrices. The transfer matrix for a particular lattice model is given by choosing a representation of one of the  $\mathcal{T}^{(n)}(z)$  over a particular ‘quantum space’. For the homogeneous  $N$  site 6-vertex model, the quantum space is the  $N$ -fold tensor product  $V_1^{(1)} \otimes V_1^{(1)} \otimes \cdots \otimes V_1^{(1)}$ , and the transfer matrix of the lattice model is

$$T^{(1)}(z) = (\pi_1^{(1)} \otimes \pi_1^{(1)} \otimes \cdots \otimes \pi_1^{(1)})\mathcal{T}^{(1)}(z).$$

Let us now consider how functional relations among the  $\mathcal{T}^{(n)}(z)$  arise. The starting point is to note that tensor products of the  $V_z^{(n)}$  have the following structure.

**Proposition 1.1.**

- (a)  $V_{zq^{n+1}}^{(n)} \otimes V_z^{(1)}$  has a unique proper  $U_q(\widehat{\mathfrak{sl}}_2)$  submodule  $V_{zq^{n+2}}^{(n-1)}$ , and furthermore  $V_{zq^{n+1}}^{(n)} \otimes V_z^{(1)} / V_{zq^{n+2}}^{(n-1)} \simeq V_{zq^n}^{(n+1)}$ ,
- (b)  $V_z^{(1)} \otimes V_{zq^{n+1}}^{(n)}$  has a unique proper  $U_q(\widehat{\mathfrak{sl}}_2)$  submodule  $V_{zq^n}^{(n+1)}$ , and furthermore  $V_z^{(1)} \otimes V_{zq^{n+1}}^{(n)} / V_{zq^n}^{(n+1)} \simeq V_{zq^{n+2}}^{(n-1)}$ .

This proposition is a specialization of the more general tensor product theorem due to Chari and Pressley [23]. Now, if  $\Delta$  denotes the coproduct of  $U_q(\widehat{\mathfrak{sl}}_2)$ , then the following is a consequence of the defining properties of  $\mathcal{R}$ :

$$\left(\pi_{zq^{n+2}}^{(n)} \otimes \pi_z^{(1)} \otimes 1\right) (\Delta \otimes 1) \mathcal{R} = \left(\pi_{zq^{n+2}}^{(n)} \otimes \pi_z^{(1)} \otimes 1\right) \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{L}_1^{(n)}(zq^{n+2}) \mathcal{L}_2^{(1)}(z).$$

If we take the trace of both sides of this equation over  $V_{zq^{n+1}}^{(n)} \otimes V_z^{(1)}$ , and use part (a) of proposition 1.1 as well as the property of the trace given by proposition 3.1 in order to rewrite the lhs, we arrive at the functional relation

$$\mathcal{T}^{(n-1)}(zq^{n+2}) + \mathcal{T}^{(n+1)}(zq^n) = \mathcal{T}^{(n)}(zq^{n+2}) \mathcal{T}^{(1)}(z). \tag{1.3}$$

Similarly, by using part (b) of proposition 1.1 we obtain

$$\mathcal{T}^{(n-1)}(zq^{n+2}) + \mathcal{T}^{(n+1)}(zq^n) = \mathcal{T}^{(1)}(z) \mathcal{T}^{(n)}(zq^{n+2}). \tag{1.4}$$

Such functional relations and this approach to deriving them are discussed in many works (see for example [24]).

Baxter’s  $Q$ -operator also obeys functional relations of a rather similar form to (1.3) and (1.4) (see [9]), and in [12, 13] the authors showed that it was possible to obtain such relations by constructing  $Q$  in a manner similar to the above. In analogy with (1.2), they proposed constructing  $Q$ -operators  $Q_{\pm}(\lambda)$  as

$$Q_{\pm}(\lambda) = \text{Tr}_{V_{\pm}(\lambda)} ((\pi_{\pm} \otimes 1) \mathcal{R}) \tag{1.5}$$

where  $(\pi_{\pm}, V_{\pm}(\lambda))$  were infinite-dimensional ‘ $q$ -oscillator’ representations of the Borel sub-algebra  $U_q(b_+)$  of  $U_q(\widehat{\mathfrak{sl}}_2)$  [13].

In this paper, we consider more general infinite-dimensional representations  $(\pi_{(z, \underline{s})}, M(z, \underline{s}))$  of  $U_q(b_+)$ , parametrized in terms of a spectral parameter  $z$  and a vector  $\underline{s} = (s_0, s_1, s_2) \in \mathbb{C}^3$ . Following [12, 13], we use them to define a  $Q$ -operator

$$\mathcal{Q}(z, \underline{s}) = \text{Tr}_{M(z, \underline{s})} ((\pi_{(z, \underline{s})} \otimes 1) \mathcal{R}).$$

We then consider tensor products of  $M(z, \underline{s})$  with  $V_z^{(1)}$  and using the result expressed in proposition 2.2 go on to derive generalized  $T$ - $Q$  relations

$$\mathcal{T}^{(1)}(z) \mathcal{Q}(z, \underline{s}) = \mathcal{Q}(z, \underline{s}) \mathcal{T}^{(1)}(z) = \mathcal{Q}(zq^2, \underline{s}^+) + \mathcal{Q}(zq^{-2}, \underline{s}^-). \tag{1.6}$$

These relations involve the shifted vectors  $\underline{s}^{\pm} = (q^{\pm 1} s_0, s_1, q^{\pm 2} s_2)$ . After a particular specialization of the vector  $\underline{s}$ , we use the appropriate representation of  $\mathcal{Q}(z, \underline{s})$  to construct an explicit form of the  $Q$ -operator for the 6-vertex model (see (5.5)). This construction works for all diagonal blocks of  $\mathcal{Q}(z, \underline{s})$ . Furthermore, the ‘spin zero’ central block of this operator coincides, up to an overall divergent factor, with Baxter’s explicit expression for this block given by equation (101) of [6] (Baxter’s construction yields an explicit expression for this block only).

The layout of the paper is as follows: In section 2, we define the infinite-dimensional representation  $M(z, \underline{s})$  of  $U_q(b_+)$  and give proposition 2.2 concerning its tensor products with  $V_z^{(1)}$ . In section 3, we define a generalized  $Q$ -operator  $\mathcal{Q}(z, \underline{s})$  and derive the  $T$ - $Q$  relations (1.6). We show that  $\mathcal{T}^{(1)}(z')$  and  $\mathcal{Q}(z, \underline{s})$  commute. In section 4, we give the

explicit form,  $\overline{Q}(z, \underline{s})$ , of our  $Q$ -operator for the 6-vertex model on a lattice with  $N$  sites, and show how the coefficients arise in the  $T$ - $Q$  relations (i.e. the coefficients  $\prod_{i=1}^N \phi_1(z, \underline{s}, w_i)$  and  $\prod_{i=1}^N \phi_2(z, \underline{s}, w_i)$  in (4.17)). We also discuss the commutation relations of  $\overline{Q}(z, \underline{s})$  and  $\overline{Q}(z', \underline{s}')$ . In section 5, we give the explicit form (5.5) of  $\overline{Q}(z, \underline{s})$  for the 6-vertex for a particular specialization of the parameter  $\underline{s}$ . We give the connection with Baxter's explicit expression for the central block. Finally, in section 6, we make some observations about our construction and discuss some possible avenues of work for the future.

## 2. Infinite-dimensional representations of $U_q(b_+)$

In this section, we define a level-zero representation  $M(z, \underline{s})$  of the Borel subalgebra  $U_q(b_+)$  of  $U'_q(\widehat{\mathfrak{sl}}_2)$ . We then consider the tensor products of  $M(z, \underline{s})$  with the spin-1/2  $U'_q(\widehat{\mathfrak{sl}}_2)$  evaluation module  $V_z^{(1)}$ . We also give the restrictions of  $\underline{s}$  for which  $M(z, \underline{s})$  reduces to a  $q$ -oscillator representation.

### 2.1. Definition of $M(z, \underline{s})$

First, let us recall the definition of  $U'_q(\widehat{\mathfrak{sl}}_2)$  and  $V_z^{(n)}$  (see, for example, [25] for an introduction to quantum affine algebras).  $U'_q(\widehat{\mathfrak{sl}}_2)$  is the associative algebra over  $\mathbb{C}$  generated by the letters  $e_i, f_i, t_i, t_i^{-1}$ , with  $i \in \{0, 1\}$ , and with relations

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} \tag{2.1}$$

$$t_i e_i t_i^{-1} = q^2 e_i \quad t_i e_j t_i^{-1} = q^{-2} e_j \quad (i \neq j) \tag{2.2}$$

$$t_i f_i t_i^{-1} = q^{-2} f_i \quad t_i f_j t_i^{-1} = q^2 f_j \quad (i \neq j) \tag{2.3}$$

$$e_i e_j^3 - [3] e_j e_i e_j^2 + [3] e_j^2 e_i e_j - e_j^3 e_i = 0 \quad (i \neq j) \tag{2.4}$$

$$f_i f_j^3 - [3] f_j f_i f_j^2 + [3] f_j^2 f_i f_j - f_j^3 f_i = 0 \quad (i \neq j). \tag{2.5}$$

We use the coproduct  $\Delta : U'_q(\widehat{\mathfrak{sl}}_2) \rightarrow U'_q(\widehat{\mathfrak{sl}}_2) \otimes U'_q(\widehat{\mathfrak{sl}}_2)$  given by

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i \quad \Delta(t_i) = t_i \otimes t_i.$$

Note, that the prime on  $U'_q(\widehat{\mathfrak{sl}}_2)$  indicates that we are not including a derivation in the definition. In this paper, we consider  $U'_q(\widehat{\mathfrak{sl}}_2)$  and its representations at generic values of  $q$  (i.e.,  $q$  is not a root of unity).

The  $U'_q(\widehat{\mathfrak{sl}}_2)$  evaluation module  $V_z^{(n)}$ ,  $n \in \mathbb{Z}_{>0}$ , is defined in terms of basis vectors  $v_j^{(n)} \otimes z^m$  with  $j \in \{0, 1, \dots, n\}$  and  $m \in \mathbb{Z}$ . The  $U'_q(\widehat{\mathfrak{sl}}_2)$  action is given by

$$\begin{aligned} e_1 \left( v_j^{(n)} \otimes z^m \right) &= [j] \left( v_{j-1}^{(n)} \otimes z^m \right) & f_1 \left( v_j^{(n)} \otimes z^m \right) &= [n - j] \left( v_{j+1}^{(n)} \otimes z^m \right) \\ t_1 \left( v_j^{(n)} \otimes z^m \right) &= q^{n-2j} \left( v_j^{(n)} \otimes z^m \right) \end{aligned} \tag{2.6}$$

$$e_0 \sim (1 \otimes z) f_1 \quad f_0 \sim (1 \otimes z^{-1}) e_1 \quad t_0 \sim t_1^{-1}.$$

We use  $(\pi_z^{(n)}, V_z^{(n)})$  to denote the spin- $n/2$  evaluation representation consisting of the module  $V_z^{(n)}$  and the associated map  $\pi_z^{(n)} : U'_q(\widehat{\mathfrak{sl}}_2) \rightarrow \text{End}(V_z^{(n)})$ .

The algebra  $U_q(b_+)$  is defined as the Borel subalgebra of  $U'_q(\widehat{\mathfrak{sl}}_2)$  generated by  $e_1, e_0, t_1, t_0$ , and  $U_q(b_-)$  is defined to be the Borel subalgebra generated by  $f_1, f_0, t_1, t_0$ . Suppose we set out to define a  $U_q(b_+)$  module in terms of basis vectors  $|j\rangle$ ,  $j \in \mathbb{Z}$ , in the following way:

$$e_1 |j\rangle = |j - 1\rangle \quad e_0 |j\rangle = \gamma_j |j + 1\rangle \quad t_1 |j\rangle = s_0 q^{-2j} |j\rangle \tag{2.7}$$

where  $\gamma_j$  and  $s_0$  are, as yet, unknown coefficients (note that we can always absorb an additional coefficient in the  $e_1|j\rangle = |j-1\rangle$  relation into a normalization of the basis vectors). Then for consistency with the Serre relations (2.4),  $\gamma_j$  must satisfy

$$\gamma_{j-3} - [3]\gamma_{j-2} + [3]\gamma_{j-1} - \gamma_j = 0. \tag{2.8}$$

The general solution of this recursion relation is

$$\gamma_j = r + s_1q^{2j} + s_2q^{-2j} \tag{2.9}$$

where  $r, s_1, s_2$  are arbitrary constants. Thus we can specify such a  $U_q(b_+)$  module by giving the four parameters  $r, s_0, s_1, s_2$ . In fact, we choose to write  $r$  in terms of  $s_1, s_2$  and a new parameter  $z$ . We make the following definition.

**Definition 2.1.**  $M(z, \underline{s})$  is a  $U_q(b_+)$  module specified in terms of basis vectors  $|j\rangle, j \in \mathbb{Z}, z \in \mathbb{C} \setminus \{0\}$  and a vector  $\underline{s} = (s_0, s_1, s_2) \in \mathbb{C}^3$ . The  $U_q(b_+)$  action is given by

$$e_1|j\rangle = |j-1\rangle \quad e_0|j\rangle = d_j(z, s_1, s_2)|j+1\rangle \quad t_1|j\rangle = s_0q^{-2j}|j\rangle \quad t_0 \sim t_1^{-1}$$

where

$$d_j(z, s_1, s_2) \equiv s_1s_2 \frac{(q - q^{-1})^2}{z} + \frac{z}{(q - q^{-1})^2} + s_1q^{2j} + s_2q^{-2j}.$$

We use the notation  $(\pi_{(z, \underline{s})}, M(z, \underline{s}))$  to indicate the representation consisting of the  $U_q(b_+)$  module  $M(z, \underline{s})$  and the associated map  $\pi_{(z, \underline{s})} : U_q(b_+) \rightarrow \text{End}(M(z, \underline{s}))$ .

### 2.2. The tensor product structure

Let us consider tensor products of  $M(z, \underline{s})$  and  $V_z^{(1)}$  as  $U_q(b_+)$  modules. We have the following proposition.

**Proposition 2.2.** If  $\underline{s}^\pm \equiv (q^{\pm 1}s_0, s_1, q^{\pm 2}s_2)$  then

- (a)  $M(z, \underline{s}) \otimes V_z^{(1)}$  has a  $U_q(b_+)$  submodule  $M(zq^2, \underline{s}^+)$ , and  $(M(z, \underline{s}) \otimes V_z^{(1)}) / M(zq^2, \underline{s}^+) \simeq M(zq^{-2}, \underline{s}^-)$ ,
- (b)  $V_z^{(1)} \otimes M(z, \underline{s})$  has a  $U_q(b_+)$  submodule  $M(zq^{-2}, \underline{s}^-)$ , and  $(V_z^{(1)} \otimes M(z, \underline{s})) / M(zq^{-2}, \underline{s}^-) \simeq M(zq^2, \underline{s}^+)$ .

**Proof.** Let us first prove (a). Define  $A_j \equiv (a_j|j\rangle \otimes v_0^{(1)} + |j-1\rangle \otimes v_1^{(1)}) \in M(z, \underline{s}) \otimes V_z^{(1)}$ , where the coefficient  $a_j$  is as yet undetermined. Clearly we have  $t_1A_j = s_0^+q^{-2j}A_j$  where  $s_0^+ = qs_0$ . The condition that  $e_1A_j = A_{j-1}$  for all  $j \in \mathbb{Z}$  is equivalent to

$$a_j + s_0q^{2(1-j)} = a_{j-1}. \tag{2.10}$$

The condition that  $e_0A_j = \kappa_jA_{j+1}$  for some coefficient  $\kappa_j$  and for all  $j \in \mathbb{Z}$  is equivalent to

$$a_jd_j(z, s_1, s_2) = a_{j+1}\kappa_j \tag{2.11}$$

and

$$a_jq^{2j}s_0^{-1}z + d_{j-1}(z, s_1, s_2) = \kappa_j \tag{2.12}$$

where the function  $d_j(z, s_1, s_2)$  is specified in definition 2.1. Solving equations (2.10)–(2.12) gives

$$a_j = -\frac{s_0s_1(1-q^2)}{zq^2} - \frac{s_0q^{2(1-j)}}{(1-q^2)} \tag{2.13}$$

$$\kappa_j = d_j(zq^2, s_1^+, s_2^+) \tag{2.14}$$

where  $s_0^+, s_1^+$  and  $s_2^+$  are the components of  $\underline{s}^+$ . In this way, we have shown that when (2.13) holds,  $A_j$  are basis vectors of a submodule isomorphic to  $M(zq^2, \underline{s}^+)$ .

Now consider  $B_j \equiv |j + 1\rangle \otimes v_0^{(1)} \in M(z, \underline{s}) \otimes V_z^{(1)}$ . We immediately have that

$$t_1 B_j = s_0 q^{-1} q^{-2j} B_j = s_0^- q^{-2j} B_j \quad \text{and} \quad e_1 B_j = B_{j-1}.$$

It is also simple to establish that

$$e_0 B_j = zq^{2(j+1)} s_0^{-1} A_{j+2} + d_j(zq^{-2}, s_1^-, s_2^-) B_{j+1}.$$

Hence we have that  $(M(z, \underline{s}) \otimes V_z^{(1)}) / M(zq^2, \underline{s}^+) \simeq M(zq^{-2}, \underline{s}^-)$ .

The proof of (b) is very similar, the only significant differences are that the analogue of the vector  $A_j$  (which is now in the submodule  $M(zq^{-2}, \underline{s}^-)$ ) is of the form

$$A_j = a_j v_0^{(1)} \otimes |j + 1\rangle + q^j v_1^{(1)} \otimes |j\rangle$$

for some  $a_j$ , and the analogue of  $B_j$  (now in the quotient module  $M(zq^2, \underline{s}^+)$ ) takes the form  $q^{-j} v_0^{(1)} \otimes |j\rangle$ . □

### 2.3. The $q$ -oscillator case

Those  $U_q(b_+)$  representations on which either  $(e_0 e_1 - q^2 e_1 e_0)$  or  $(e_1 e_0 - q^2 e_0 e_1)$  acts as a constant are referred to as  $q$ -oscillator representations [26]. In terms of the action (2.7), these requirements become either

$$\gamma_{j-1} - q^2 \gamma_j = \text{constant} \tag{2.15}$$

or

$$\gamma_j - q^2 \gamma_{j-1} = \text{constant} \tag{2.16}$$

respectively. Either of these conditions separately implies (2.8). The general solutions of (2.15) and (2.16) are

$$\gamma_j = r + s_2 q^{-2j}$$

and

$$\gamma_j = r + s_1 q^{2j}$$

respectively, where  $r, s_1, s_2$  are arbitrary constants.

Thus the specializations  $M(z, s_0, 0, s_2)$  and  $M(z, s_0, s_1, 0)$  are both  $q$ -oscillator representations. Connecting with the notation  $V_{\pm}(\lambda)$  notation of [13]:  $M(\lambda, s_0, 0, s_2)$  is a representation of the type  $V_+(\lambda)$  and  $M(\lambda, s_0, s_1, 0)$  is a representation of the type  $V_-(\lambda)$ .<sup>2</sup> The two operators  $Q_{\pm}(\lambda)$  of [13] (see our (1.5)) are obtained by specializing our construction (3.4) accordingly.

### 3. The generalized $Q$ -operator

In this section, we will construct a  $U_q(b_-)$  valued  $Q$ -operator  $\mathcal{Q}(z, \underline{s})$  in terms of the universal  $R$ -matrix and the  $U_q(b_+)$  module  $M(z, \underline{s})$  defined in the last section. We will go on to show how generalized  $T$ - $Q$  relations arise as a consequence of proposition 2.2. We will then consider representations of  $\mathcal{Q}(z, \underline{s})$ .

<sup>2</sup> In [13], the notation  $V_{\pm}(\lambda)$  seems to refer originally to a class of representations; but in appendix B,  $V_+(\lambda)$  refers to a specific representation, and in this case we have  $V_+(\lambda) \simeq M(\lambda, 1, 0, 0)$ .

### 3.1. The operator $\mathcal{Q}(z, \underline{s})$

We make use of the universal  $R$ -matrix of  $U_q(\widehat{\mathfrak{sl}}_2)$ , which we denote by  $\mathcal{R} \in U_q(b_+) \otimes U_q(b_-)$ . The definition of  $\mathcal{R}$  can be found in [27]; here we need only the properties

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \quad (1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \quad (3.1)$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \quad (3.2)$$

where as usual  $\mathcal{R}_{12} = \mathcal{R} \otimes 1$  etc.

Then using the representations defined in section 2, we make the following definitions:

$$\mathcal{L}(z) = (\pi_z^{(1)} \otimes 1) \mathcal{R} \in \text{End}(V_z^{(1)}) \otimes U_q(b_-)$$

$$\mathcal{W}(z, \underline{s}) = (\pi_{(z, \underline{s})} \otimes 1) \mathcal{R} \in \text{End}(M(z, \underline{s})) \otimes U_q(b_-).$$

By taking the trace we go on to define

$$\mathcal{T}(z) = \text{Tr}_{V_z^{(1)}}(\mathcal{L}(z)) \in U_q(b_-) \quad (3.3)$$

$$\mathcal{Q}(z, \underline{s}) = \text{Tr}_{M(z, \underline{s})}(\mathcal{W}(z, \underline{s})) \in U_q(b_-). \quad (3.4)$$

The operators  $\mathcal{L}(z)$  and  $\mathcal{T}(z)$  are the familiar monodromy matrix and transfer matrix of the QISM (although these terms are perhaps more commonly reserved for representations of these algebraic objects on particular quantum spaces). The operator  $\mathcal{Q}(z, \underline{s})$  is our generalized  $Q$ -operator.

### 3.2. $T$ - $Q$ relations

Our starting point in the derivation of  $T$ - $Q$  relations is the following simple proposition.

**Proposition 3.1.** *If  $\mathcal{A}$  is an associative algebra,  $X \in \mathcal{A}$ , and  $A, B, C$  are finite-dimensional  $\mathcal{A}$  modules which form an exact sequence  $0 \rightarrow B \rightarrow A \rightarrow C \rightarrow 0$ , then  $\text{Tr}_A(X) = \text{Tr}_B(X) + \text{Tr}_C(X)$ .*

A proof is given in appendix A. To proceed, let us consider the expression

$$(\pi_{(z, \underline{s})} \otimes \pi_z^{(1)} \otimes 1) (\Delta \otimes 1)\mathcal{R} \in \text{End}(M(z, \underline{s})) \otimes \text{End}(V_z^{(1)}) \otimes U_q(b_-). \quad (3.5)$$

Using the first property in (3.1), we arrive at

$$(\pi_{(z, \underline{s})} \otimes \pi_z^{(1)} \otimes 1) (\Delta \otimes 1)\mathcal{R} = (\pi_{(z, \underline{s})} \otimes \pi_z^{(1)} \otimes 1) \mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{W}_1(z, \underline{s})\mathcal{L}_2(z). \quad (3.6)$$

Suppose we assume that proposition 3.1 also holds for the *infinite-dimensional*  $U_q(b_+)$  modules involved in the exact sequence

$$0 \rightarrow M(zq^2, \underline{s}^+) \rightarrow M(z, \underline{s}) \otimes V_z^{(1)} \rightarrow M(zq^{-2}, \underline{s}^-) \rightarrow 0$$

whose existence is equivalent to part (a) of proposition 2.2. In this case, we will have

$$\text{Tr}_{M(z, \underline{s}) \otimes V_z^{(1)}}(\Delta(X)) = \text{Tr}_{M(zq^2, \underline{s}^+)}(X) + \text{Tr}_{M(zq^{-2}, \underline{s}^-)}(X) \quad (3.7)$$

for  $X \in U_q(b_+)$ . Taking the trace over  $M(z, \underline{s}) \otimes V_z^{(1)}$  of both sides of (3.6) and using (3.7) to rewrite the lhs (as well as the definitions (3.3) and (3.4)) yield

$$\mathcal{Q}(zq^2, \underline{s}^+) + \mathcal{Q}(zq^{-2}, \underline{s}^-) = \mathcal{Q}(z, \underline{s})\mathcal{T}(z) \in U_q(b_-).$$

A similar argument, which now relies on part (b) of proposition 2.2 gives

$$\mathcal{Q}(zq^{-2}, \underline{s}^-) + \mathcal{Q}(zq^2, \underline{s}^+) = \mathcal{T}(z)\mathcal{Q}(z, \underline{s}) \in U_q(b_-).$$

Thus we arrive at the  $T$ - $Q$  relations

$$\mathcal{T}(z)\mathcal{Q}(z, \underline{s}) = \mathcal{Q}(z, \underline{s})\mathcal{T}(z) = \mathcal{Q}(zq^2, \underline{s}^+) + \mathcal{Q}(zq^{-2}, \underline{s}^-). \quad (3.8)$$

In sections 4.3 and 5, we discuss the meaning of the infinite-dimensional trace involved in the definition of  $\mathcal{Q}(z)$ , and describe various checks of the  $T$ - $Q$  relations (3.8) that arise as a consequence of the assumption about the validity of the extension of proposition 3.1.



### 3.3. Commutation relations

The Yang–Baxter relation (3.2) is an equality between elements of  $U_q(b_+) \otimes U'_q(\widehat{\mathfrak{sl}}_2) \otimes U_q(b_-)$ . Acting on both sides with  $\pi_{(z, \underline{s})} \otimes \pi_{z'}^{(1)} \otimes 1$  (where  $z' \in \mathbb{C}$  is arbitrary) gives

$$W(z, \underline{s}; z') \mathcal{W}(z, \underline{s}) \mathcal{L}(z') = \mathcal{L}(z') \mathcal{W}(z, \underline{s}) W(z, \underline{s}; z') \in \text{End}_{M(z, \underline{s})} \otimes \text{End}_{V_{z'}^{(1)}} \otimes U_q(b_-)$$

where  $W(z, \underline{s}; z') \equiv (\pi_{(z, \underline{s})} \otimes \pi_{z'}^{(1)}) \mathcal{R}$ . Multiplying on the right by  $W(z, \underline{s}; z')^{-1}$ , taking the trace over  $M(z, \underline{s}) \otimes V_{z'}^{(1)}$ , and using the definitions (3.3) and (3.4) we obtain the commutation relations

$$[Q(z, \underline{s}), \mathcal{T}(z')] = 0. \quad (3.9)$$

It is an obvious next step to attempt to repeat this argument and act with  $\pi_{(z, \underline{s})} \otimes \pi_{(z', \underline{s}')} \otimes 1$  on (3.2) in order to derive commutation relations involving  $Q(z, \underline{s})$  and  $Q(z', \underline{s}')$ . However, such an argument *fails* for the simple reason that  $M(z', \underline{s}')$  is only a  $U_q(b_+)$  module and not a  $U'_q(\widehat{\mathfrak{sl}}_2)$  module. In fact, as discussed in detail in section 4.4,  $Q(z, \underline{s})$  and  $Q(z', \underline{s}')$  do *not* commute for general  $\underline{s}$  and  $\underline{s}'$ .

### 3.4. Representations of $Q(z, \underline{s})$

We have constructed both  $\mathcal{T}(z)$  and  $Q(z, \underline{s})$  as  $U_q(b_-)$  valued objects. Constructing the operators corresponding to a particular lattice model simply involves choosing a representation of  $U_q(b_-)$  (i.e., choosing a  $U_q(b_-)$  module  $V_{qu}$  and the associated map  $\pi_{qu} : U_q(b_-) \rightarrow \text{End}(V_{qu})$ ). Such a representation is referred to as the quantum space in the language of the QISM. If we define

$$T(z) \equiv \pi_{qu}(\mathcal{T}(z)) \in \text{End}(V_{qu}) \quad \text{and} \quad Q(z, \underline{s}) \equiv \pi_{qu}(Q(z, \underline{s})) \in \text{End}(V_{qu})$$

it then follows from (3.8) and (3.9) that we have the  $T$ – $Q$  relations

$$T(z)Q(z, \underline{s}) = Q(z, \underline{s})T(z) = Q(zq^2, \underline{s}^+) + Q(zq^{-2}, \underline{s}^-) \quad (3.10)$$

and the commutation relations

$$[Q(z, \underline{s}), \mathcal{T}(z')] = 0.$$

If we choose our quantum space to be equal to  $V_1 \otimes V_2 \otimes \cdots \otimes V_N$ , as will be the case for lattice models, then it follows from (3.1) that we have

$$T(z) = \text{Tr}_{V_z^{(1)}}(L_N(z)L_{N-1}(z) \cdots L_1(z))$$

and

$$Q(z, \underline{s}) = \text{Tr}_{M(z, \underline{s})}(W_N(z, \underline{s})W_{N-1}(z, \underline{s}) \cdots W_1(z, \underline{s}))$$

where

$$L_i(z) \equiv (1 \otimes \pi_{V_i}) \mathcal{L}(z) \in \text{End}(V_z^{(1)}) \otimes \text{End}(V_i)$$

and

$$W_i(z, \underline{s}) \equiv (1 \otimes \pi_{V_i}) \mathcal{W}(z, \underline{s}) \in \text{End}(M(z, \underline{s})) \otimes \text{End}(V_i).$$

Let us comment on the connection between our  $T$ – $Q$  relations (3.10) and Baxter’s  $T$ – $Q$  relations [9]. Two differences are immediately apparent. Firstly, there are more parameters in our relation;  $Q(z, \underline{s})$  depends upon the spectral parameter  $z$  and  $\underline{s} = (s_0, s_1, s_2) \in \mathbb{C}^3$  as well as the parameter  $q$  in the universal  $R$ -matrix and all parameters associated with the quantum space  $V_{qu}$ . In section 5, we compute our  $Q$ -matrix explicitly for the 6-vertex model, and

show how with a particular specialization it is related to Baxter's  $Q$ -matrix. Secondly, the coefficients multiplying the  $Q(zq^2, \underline{s}^+)$  and  $Q(zq^{-2}, \underline{s}^-)$  on the rhs of (3.10) are 1, unlike in Baxter's case. This is a trivial point associated with the normalization of the  $Q$ -matrix that we have used in this section. The coefficients will reappear in section 4.2, when we choose to normalize our  $Q(z, \underline{s})$  in a more practical way.

#### 4. Properties of $Q$ for the 6-vertex model

In this section, we will consider the operator  $W_i(z, \underline{s})$  when the  $V_i$  appearing in the quantum space  $V_1 \otimes V_2 \otimes \cdots \otimes V_N$  is equal to  $V_{w_i}^{(1)}$ , i.e., in the 6-vertex model case. We will determine the action of  $W_i(z, \underline{s})$  on the space  $M(z, \underline{s}) \otimes V_{w_i}^{(1)}$  up to a multiplicative constant, and show how the coefficients appearing on the rhs of the  $T$ - $Q$  relation depend on this normalization factor. Finally, we will address the problem of the commutativity of  $Q$ -operators.

##### 4.1. Definition of $W_i$ and $Q$ for the 6-vertex model

Let us consider the operator

$$W(z, \underline{s}; w) \equiv (\pi_{(z, \underline{s})} \otimes \pi_w^{(1)}) \mathcal{R} \in \text{End}(M(z, \underline{s})) \otimes \text{End}(V_w^{(1)}). \quad (4.1)$$

$W(z, \underline{s}; w_i)$  is the operator  $W_i(z, \underline{s})$  associated with the 6-vertex model with local inhomogeneity parameter  $w_i$ .

In order to determine  $W(z, \underline{s}; w)$  explicitly, definition (4.1) is rather inconvenient, because it involves the unwieldy universal  $R$ -matrix. It is easier to construct an operator  $\overline{W}(z, \underline{s}; w) \in \text{End}(M(z, \underline{s})) \otimes \text{End}(V_w^{(1)})$  which satisfies the properties

$$\overline{W}(z, \underline{s}; w) (\pi_{(z, \underline{s})} \otimes \pi_w^{(1)}) \Delta(t_i) = (\pi_{(z, \underline{s})} \otimes \pi_w^{(1)}) \Delta'(t_i) \overline{W}(z, \underline{s}; w) \quad (4.2)$$

$$\overline{W}(z, \underline{s}; w) (\pi_{(z, \underline{s})} \otimes \pi_w^{(1)}) \Delta(e_i) = (\pi_{(z, \underline{s})} \otimes \pi_w^{(1)}) \Delta'(e_i) \overline{W}(z, \underline{s}; w). \quad (4.3)$$

Since  $M(z, \underline{s}) \otimes V_w^{(1)}$  is an irreducible  $U_q(b_+)$  module for generic  $z, \underline{s}$  and  $w$ , it follows that an operator satisfying properties (4.2)–(4.3) is unique up to a multiplicative constant. By definition,  $W(z, \underline{s}; w)$  also satisfies (4.2)–(4.3) and so will be proportional to  $\overline{W}(z, \underline{s}; w)$ .

Let us solve (4.2)–(4.3). Firstly, relation (4.2) requires that  $\overline{W}(z, \underline{s}; w)$  must be of the form

$$\overline{W}(z, \underline{s}; w)|j\rangle \otimes v_0^{(1)} = \alpha_{j,0}|j\rangle \otimes v_0^{(1)} + \beta_{j,0}|j-1\rangle \otimes v_1^{(1)} \quad (4.4)$$

$$\overline{W}(z, \underline{s}; w)|j\rangle \otimes v_1^{(1)} = \alpha_{j,1}|j\rangle \otimes v_1^{(1)} + \beta_{j,1}|j+1\rangle \otimes v_0^{(1)} \quad (4.5)$$

where  $\alpha_{j,0}, \alpha_{j,1}, \beta_{j,0}$  and  $\beta_{j,1}$  are arbitrary coefficients. Then, (4.3) is satisfied if and only if

$$\beta_{j-1,0} = q^{-1}\beta_{j,0}$$

$$\alpha_{j-1,0} = q\alpha_{j,0} + \beta_{j,0}$$

$$\alpha_{j-1,1} + s_0q^{-2j}\beta_{j,0} = q^{-1}\alpha_{j,1}$$

$$\beta_{j-1,1} + s_0q^{-2j}\alpha_{j,0} = \alpha_{j,1} + q\beta_{j,1}$$

$$d_j(z, s_1, s_2)\alpha_{j+1,0} + s_0^{-1}q^{2j}w\beta_{j,1} = q^{-1}d_j(z, s_1, s_2)\alpha_{j,0}$$

$$d_j(z, s_1, s_2)\beta_{j+1,0} + s_0^{-1}q^{2j}w\alpha_{j,1} = w\alpha_{j,0} + qd_{j-1}(z, s_1, s_2)\beta_{j,0}$$

$$d_j(z, s_1, s_2)\alpha_{j+1,1} = qd_j(z, s_1, s_2)\alpha_{j,1} + w\beta_{j,1}$$

$$d_j(z, s_1, s_2)\beta_{j+1,1} = q^{-1}d_{j+1}(z, s_1, s_2)\beta_{j,1}$$

where the function  $d_j(z, s_1, s_2)$  is specified in definition 2.1. These equations have the general solution

$$\alpha_{j,0} = \left( \frac{s_2}{w} q(1 - q^2)q^{-j} - \frac{q^j}{q - q^{-1}} \right) \rho \quad \alpha_{j,1} = \left( \frac{s_0 s_1}{w} q(q^{-2} - 1)q^j - s_0 \frac{q^{-j}}{q - q^{-1}} \right) \rho \tag{4.6}$$

$$\beta_{j,0} = q^j \rho \quad \beta_{j,1} = s_0 \frac{q^{-j}}{w} d_j(z, s_1, s_2) \rho$$

where  $\rho$  is an arbitrary constant.

Let us choose the normalization constant  $\rho$  to be an as yet unspecified function  $\rho(z, \underline{s}, w)$ , and define the  $Q$ -operator in terms of  $\overline{W}(z, \underline{s}, w)$  (given by (4.4), (4.5) and (4.6)) by

$$\overline{Q}(z, \underline{s}) = \text{Tr}_{M(z, \underline{s})}(\overline{W}(z, \underline{s}; w_N) \overline{W}(z, \underline{s}; w_{N-1}) \cdots \overline{W}(z, \underline{s}; w_1)). \tag{4.7}$$

#### 4.2. Normalized $T$ - $Q$ relations for the 6-vertex model

Since we are dealing with the 6-vertex model, we have to consider the Lax operator  $L_i(z)$  of section 3 in the case when the quantum space is  $V_i = V_{w_i}^{(1)}$ . This is given by the matrix

$$R(z/w_i) = (\pi_z^{(1)} \otimes \pi_{w_i}^{(1)}) \mathcal{R} \tag{4.8}$$

acting on  $V_z^{(1)} \otimes V_{w_i}^{(1)}$ . The matrix  $R(z/w)$  is proportional to the normalized matrix  $\overline{R}(z/w)$  given by

$$\begin{aligned} \overline{R}(z/w)v_0^{(1)} \otimes v_0^{(1)} &= v_0^{(1)} \otimes v_0^{(1)} \\ \overline{R}(z/w)v_0^{(1)} \otimes v_1^{(1)} &= \frac{q(1 - \frac{z}{w})}{1 - q^2 \frac{z}{w}} v_0^{(1)} \otimes v_1^{(1)} + \frac{\frac{z}{w}(1 - q^2)}{1 - q^2 \frac{z}{w}} v_1^{(1)} \otimes v_0^{(1)} \\ \overline{R}(z/w)v_1^{(1)} \otimes v_0^{(1)} &= \frac{q(1 - \frac{z}{w})}{1 - q^2 \frac{z}{w}} v_1^{(1)} \otimes v_0^{(1)} + \frac{1 - q^2}{1 - q^2 \frac{z}{w}} v_0^{(1)} \otimes v_1^{(1)} \\ \overline{R}(z/w)v_1^{(1)} \otimes v_1^{(1)} &= v_1^{(1)} \otimes v_1^{(1)}. \end{aligned} \tag{4.9}$$

We define the normalized transfer matrix of the 6-vertex model by

$$\overline{T}(z) = \text{Tr}_{V_z^{(1)}}(\overline{R}(z/w_N) \overline{R}(z/w_{N-1}) \cdots \overline{R}(z/w_1)). \tag{4.10}$$

We are going to derive the coefficients in the  $T$ - $Q$  relation associated with the above normalization of  $\overline{T}(z)$  and  $\overline{Q}(z, \underline{s})$ . As a preliminary, let us introduce a little more notation: we use  $|j\rangle^\pm$  to denote the basis vectors in  $M(zq^{\pm 2}, \underline{s}^\pm)$ ;  $\iota$  to denote the embedding

$$\begin{aligned} \iota : M(zq^2, \underline{s}^+) &\rightarrow M(z, \underline{s}) \otimes V_z^{(1)} \\ |j\rangle^+ &\mapsto A_j \end{aligned}$$

and  $\pi$  to denote the projection

$$\begin{aligned} \pi : M(z, \underline{s}) \otimes V_z^{(1)} &\rightarrow M(zq^{-2}, \underline{s}^-) \\ B_j &\mapsto |j\rangle^-. \end{aligned}$$

$A_j$  and  $B_j$  are as defined in the proof of proposition 2.2.

The coefficients of the  $T$ - $Q$  relation are then obtained from the action of  $\overline{W}(z, \underline{s}; w) \overline{R}(z/w)$  on the space  $M(z, \underline{s}) \otimes V_z^{(1)} \otimes V_w^{(1)}$ . By direct calculation we find

$$\overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) A_j \otimes v_0^{(1)} = \phi_1(z, \underline{s}, w) \left[ \alpha_{j,0}^+ A_j \otimes v_0^{(1)} + \beta_{j,0}^+ A_{j-1} \otimes v_1^{(1)} \right] \tag{4.11}$$

$$\overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) A_j \otimes v_1^{(1)} = \phi_1(z, \underline{s}, w) \left[ \alpha_{j,1}^+ A_j \otimes v_1^{(1)} + \beta_{j,1}^+ A_{j+1} \otimes v_0^{(1)} \right] \tag{4.12}$$

$$\overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) B_j \otimes v_0^{(1)} = \phi_2(z, \underline{s}, w) \left[ \alpha_{j,0}^- B_j \otimes v_0^{(1)} + \beta_{j,0}^- B_{j-1} \otimes v_1^{(1)} \right] \quad (4.13)$$

$$\overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) B_j \otimes v_1^{(1)} = \phi_2(z, \underline{s}, w) \left[ \alpha_{j,1}^- B_j \otimes v_1^{(1)} + \beta_{j,1}^- B_{j+1} \otimes v_0^{(1)} \right] \quad (4.14)$$

$$+ \frac{z(1-q^2)}{w-q^2z} \beta_{j+1,0} A_{j+1} \otimes v_1^{(1)} + \frac{z(1-q^2)}{w-q^2z} \alpha_{j+1,0} A_{j+2} \otimes v_0^{(1)} \quad (4.15)$$

where

$$\phi_1(z, \underline{s}, w) \equiv \frac{\rho(z, \underline{s}, w)}{\rho(zq^2, \underline{s}^+, w)} \frac{w-z}{w-q^2z} \quad \phi_2(z, \underline{s}, w) \equiv \frac{\rho(z, \underline{s}, w)}{\rho(zq^{-2}, \underline{s}^-, w)} q. \quad (4.16)$$

$\alpha_{j,0}^\pm, \alpha_{j,1}^\pm, \beta_{j,0}^\pm, \beta_{j,1}^\pm$  are the coefficients (see relations (4.4)–(4.6)) specifying the action of  $\overline{W}(z, \underline{s}^\pm; w)$  on  $M(z, \underline{s}^\pm) \otimes V_w^{(1)}$ .

An immediate consequence of (4.11)–(4.15) is that we have

$$\begin{aligned} \overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) \iota(|j\rangle^+) \otimes v_\varepsilon^{(1)} &= \phi_1(z, \underline{s}, w) (\iota \otimes 1) \overline{W}(zq^2, \underline{s}^+; w) |j\rangle^+ \otimes v_\varepsilon^{(1)} \\ (\pi \otimes 1) \overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w) B_j \otimes v_\varepsilon^{(1)} &= \phi_2(z, \underline{s}, w) \overline{W}(zq^{-2}, \underline{s}^-; w) |j\rangle^- \otimes v_\varepsilon^{(1)} \end{aligned}$$

for  $\varepsilon \in \{0, 1\}$ . It then follows (by the appropriate modification of the argument in the proof in appendix A by the factors  $\phi_1$  and  $\phi_2$ ) that

$$\begin{aligned} \text{Tr}_{M(z, \underline{s}) \otimes V_\varepsilon^{(1)}} (\overline{W}_{13}(z, \underline{s}; w) \overline{R}_{23}(z/w)) &= \phi_1(z, \underline{s}, w) \text{Tr}_{M(zq^2, \underline{s}^+)} (\overline{W}(zq^2, \underline{s}^+; w)) \\ &+ \phi_2(z, \underline{s}, w) \text{Tr}_{M(zq^{-2}, \underline{s}^-)} (\overline{W}(zq^{-2}, \underline{s}^-; w)). \end{aligned}$$

This is the normalized  $T$ – $Q$  relation associated with a quantum space  $V_w^{(1)}$ . More generally, a modification of the above argument to the case when the quantum space is  $V_{w_1}^{(1)} \otimes V_{w_2}^{(1)} \otimes \dots \otimes V_{w_N}^{(1)}$  gives the normalized  $T$ – $Q$  relations

$$\overline{Q}(z, \underline{s}) \overline{T}(z) = \overline{T}(z) \overline{Q}(z, \underline{s}) = \left( \prod_{i=1}^N \phi_1(z, \underline{s}, w_i) \right) \overline{Q}(zq^2, \underline{s}^+) + \left( \prod_{i=1}^N \phi_2(z, \underline{s}, w_i) \right) \overline{Q}(zq^{-2}, \underline{s}^-). \quad (4.17)$$

### 4.3. Some comments on the definition of $Q$

We have defined our generalized  $Q$ -operator for the 6-vertex model by formula (4.7). Let us make some comments on this definition. First of all, we note that, due to the charge conservation property (4.2), the operator (4.7) is a block-diagonal matrix, each block connecting vectors of  $V_{w_1}^{(1)} \otimes \dots \otimes V_{w_N}^{(1)}$  containing the same number of  $v_0^{(1)}$ . Next, we remark that it follows from the explicit expression for  $\overline{W}(z, \underline{s}; w)$  given by (4.4)–(4.6), that matrix elements of the  $Q$ -operator (4.7) will have the following form:

$$\sum_{k=0}^N a_k(z, \underline{s}, w_1, \dots, w_N, q) \delta(q^k)$$

where  $a_k(z, \underline{s}, w_1, \dots, w_N, q)$  is a product of the  $\rho(z, \underline{s}, w_i)$  normalization factors with a rational function of  $z, \underline{s}, w_1, \dots, w_N, q$ , and  $\delta(q^k) \equiv \sum_{j \in \mathbb{Z}} q^{kj}$ . When  $k \neq 0$ ,  $\delta(q^k)$  is a formal series; but clearly some care is needed in interpreting the meaning of  $\delta(q^k)$  when  $k = 0$ ! The situation becomes clearer if we restrict the case  $s_1 = s_2 = 0$ : with this restriction this  $\delta(q^0)$  appears only as an overall multiplicative factor in the central block of  $Q$  connecting  $N/2$  vectors  $v_0^{(1)}$ , and thus cancels from both sides of the  $T$ – $Q$  relation. Moreover, Baxter has an explicit expression for the central block of his  $Q$ -operator [6], and in section 5 we establish

an equality between the central block of our  $Q$  (with the  $s_1 = s_2 = 0$  restriction and  $\delta(q^0)$  replaced by 1) and Baxter’s operator.

This result leads us to conjecture that the  $T$ – $Q$  relations (4.17) hold if we simply replace  $\delta(q^0)$ , wherever it appears in the matrix elements of  $\overline{Q}(z, \underline{s})$ , by a constant (say 1). As further support to this conjecture, we have checked explicitly in the case when  $N = 2$  and  $\underline{s}$  is generic, that the  $T$ – $Q$  relation holds independently of the value assigned to  $\delta(q^0)$ .

#### 4.4. Commutation relations of $Q$ for the 6-vertex model

In section 3, we have constructed a generalized  $Q$ -operator  $Q(z, \underline{s})$  satisfying the  $T$ – $Q$  relation (3.8) and commuting with  $T(z')$ . We have already remarked in section 3.3 that, unlike the commutativity with  $T(z')$ , the commutativity of  $Q(z, \underline{s})$  with  $Q(z', \underline{s}')$  cannot be shown by general algebraic arguments. Therefore, in order to deal with this fact, we have to choose particular representations for the quantum space. We will focus on the 6-vertex model, for which the  $Q$ -operator is given by  $\overline{Q}(z, \underline{s})$  (4.7).

Let us start by saying that explicit calculations performed in the case  $N = 2$  show that  $\overline{Q}(z, \underline{s})$  and  $\overline{Q}(z', \underline{s}')$  do *not* commute for general values of the parameters  $z, \underline{s}$  and  $z', \underline{s}'$ . On the other hand, we will show in the following section that such commutativity holds for general  $N$  when we restrict to the case

$$s_1 = s_2 = s'_1 = s'_2 = 0. \tag{4.18}$$

### 5. An explicit form of $Q$ for the 6-vertex model

In this section, we give a simple explicit expression for our  $Q$ -operator (4.7) for general  $N$ , when we make the specialization  $s_1 = s_2 = 0$ . We shall find that this expression is related to equation (101) of [6] (which is an expression for the central block of the  $Q$ -matrix when the number of lattice sites  $N$  is even).

First of all, let us define matrix elements of  $\overline{W}(z, \underline{s}, w)$  and  $\overline{Q}(z; s)$  of section 4.1. In this section, we will use the notation  $v_{+(-)}^{(1)}$  in place of  $v_{0(1)}^{(1)}$  in order to facilitate comparison with [6]. We define matrix elements by

$$\overline{W}(z, \underline{s}, w) |j\rangle \otimes v_{\beta}^{(1)} = \sum_{\alpha} \overline{W}(z, \underline{s}, w)_{j'\alpha}^{j\beta} |j\rangle \otimes v_{\alpha}^{(1)} \quad \text{where } j' = j + (\alpha - \beta)/2$$

$$\overline{Q}(z, \underline{s}) \left( v_{\beta_1}^{(1)} \otimes v_{\beta_2}^{(1)} \otimes \cdots \otimes v_{\beta_N}^{(1)} \right) = \sum_{\alpha_1, \alpha_2, \dots, \alpha_N} \overline{Q}(z, \underline{s})_{\alpha_1, \alpha_2, \dots, \alpha_N}^{\beta_1, \beta_2, \dots, \beta_N} \left( v_{\alpha_1}^{(1)} \otimes v_{\alpha_2}^{(1)} \otimes \cdots \otimes v_{\alpha_N}^{(1)} \right).$$

Fixing  $w_1 = \cdots = w_N = w$ , it then follows from (4.7) that we have

$$\overline{Q}(z, \underline{s})_{\alpha_1, \dots, \alpha_N}^{\beta_1, \dots, \beta_N} = \sum_{j \in \mathbb{Z}} \overline{W}(z, \underline{s}; w)_{j, \alpha_N}^{j_{N-1}, \beta_N} \overline{W}(z, \underline{s}; w)_{j_{N-1}, \alpha_{N-1}}^{j_{N-2}, \beta_{N-1}} \cdots \overline{W}(z, \underline{s}; w)_{j_1, \alpha_1}^{j, \beta_1}. \tag{5.1}$$

where  $j_k = j + \frac{1}{2}(\alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}) - \frac{1}{2}(\beta_1 + \beta_2 + \cdots + \beta_{k-1})$ .

Now let us consider the matrix elements  $\overline{W}(z, \underline{s}, w)_{j'\alpha}^{j\beta}$  given by (4.6), in the special case when  $s_1 = s_2 = 0$ . In this case, we have

$$\overline{W}(z, \underline{s}, w)_{j_+}^{j_+} = -\frac{\rho q^j}{q - q^{-1}} \quad \overline{W}(z, \underline{s}, w)_{j_-}^{j_-} = -\frac{s_0 \rho q^{-j}}{q - q^{-1}} \tag{5.2}$$

$$\overline{W}(z, \underline{s}, w)_{j_{-1-}}^{j_+} = \rho q^j \quad \overline{W}(z, \underline{s}, w)_{j_{+1+}}^{j_-} = \frac{z}{w} \frac{s_0 \rho q^{-j}}{(q - q^{-1})^2}. \tag{5.3}$$

It is a simple consequence of charge conservation (i.e. the property that  $\overline{W}(z, \underline{s}, w)_{j'\alpha}^{j\beta}$  is only nonzero for  $j' = j + (\alpha - \beta)/2$ ) that whenever we have a contribution to (5.1) of the form  $\overline{W}(z, \underline{s}, w)_{j-1-}^+$  we must also have one of the form  $\overline{W}(z, \underline{s}, w)_{j+1+}^-$ . So when computing (5.1), we can equally use the more symmetric expressions

$$\overline{W}(z, \underline{s}, w)_{j-1-}^{j+} = -\left(\frac{z}{w}\right)^{\frac{1}{2}} \frac{\sqrt{s_0 \rho} q^j}{q - q^{-1}} \quad \overline{W}(z, \underline{s}, w)_{j+1+}^{j-} = -\left(\frac{z}{w}\right)^{\frac{1}{2}} \frac{\sqrt{s_0 \rho} q^{-j}}{q - q^{-1}}.$$

Then, if we define  $\tilde{\rho} = -\frac{\sqrt{s_0 \rho}}{q - q^{-1}}$ , we can write both (5.2) and (5.3) as

$$\overline{W}(z, \underline{s}, w)_{j'\alpha}^{j\beta} = \left(\frac{z}{w}\right)^{(1-\alpha\beta)/4} s_0^{-(\alpha+\beta)/4} \tilde{\rho} q^{j\beta}. \tag{5.4}$$

Using the form (5.4), it is then straightforward to compute a simple expression for (5.1). The matrix is block diagonal, consisting of blocks for which  $\alpha_1 + \alpha_2 + \dots + \alpha_N = \beta_1 + \beta_2 + \dots + \beta_N = n$ , where  $n \in \{-N, -N + 2, \dots, N\}$ . The block labelled by  $n$  in this way has the form

$$\overline{Q}(z, \underline{s})_{\alpha_1, \dots, \alpha_N}^{\beta_1, \dots, \beta_N} = s_0^{-n/2} \delta(q^n) \left(\frac{zq}{w}\right)^{N/4} \tilde{\rho}^N q^{\frac{1}{4} \sum_{i < j} (\beta_j \alpha_i - \alpha_j \beta_i)} \left(\frac{w}{zq}\right)^{\frac{1}{4} \sum_i \alpha_i \beta_i} \tag{5.5}$$

where the function  $\delta(q^n) = \sum_{j \in \mathbb{Z}} q^{nj}$  arises from sum over  $j$  in expression (5.1).

Now, let us compare (5.5) to Baxter's  $Q$ -matrix given by equation (101) of [6] as

$$[Q_{\text{Bax}}(v)]_{\alpha_1, \dots, \alpha_N}^{\beta_1, \dots, \beta_N} = \exp \left[ \frac{1}{2} i \eta \sum_{k=1}^N \sum_{j=1}^{k-1} (\alpha_j \beta_k - \alpha_k \beta_j) + \frac{1}{2} i v \sum_{j=1}^N \alpha_j \beta_j \right]. \tag{5.6}$$

Clearly, if we identify

$$q = \exp(2i\eta) \quad \frac{w}{zq} = \exp(2iv) \tag{5.7}$$

and choose the arbitrary normalization function  $\tilde{\rho}$  to be

$$\tilde{\rho} = \tilde{\rho}(z, \underline{s}, w) = (zq/w)^{-\frac{1}{4}} \tag{5.8}$$

then we have

$$\overline{Q}(z, \underline{s})_{\alpha_1, \dots, \alpha_N}^{\beta_1, \dots, \beta_N} = s_0^{-n/2} \delta(q^n) [Q_{\text{Bax}}(v)]_{\alpha_1, \dots, \alpha_N}^{\beta_1, \dots, \beta_N}. \tag{5.9}$$

Let us emphasize, that whereas (5.6) is derived in [6] as an expression for the  $n = 0$  block of the  $Q$ -matrix, our expression (5.5) or (5.9) is valid for all blocks (the  $n = 0$  case is discussed below). It is also valid for  $N$  both even and odd. Also note that the choice (5.7) is the one required in order to identify (up to a normalization and gauge transformation) our 6-vertex model  $R$ -matrix  $\overline{R}(z/w)$  with Baxter's  $R$ -matrix given as a function of  $v$  and  $\eta$ .<sup>3</sup>

Let us now consider the  $\delta(q^n)$  term appearing in (5.9), and its meaning in the context of the  $T$ - $Q$  relations (4.17) and the commutativity with  $T$ . For this purpose, it is useful to define a new matrix without the delta function:

$$\tilde{Q}(z, \underline{s}) \equiv \overline{Q}(z, \underline{s}) / \delta(q^n) = s_0^{-n/2} Q_{\text{Bax}}(v). \tag{5.10}$$

For  $n = 0$ ,  $\delta(q^n)$  is clearly meaningless, but we conjecture that  $\tilde{Q}(z, \underline{s})$  still obeys the  $T$ - $Q$  relations (4.17) and still commutes with  $\overline{T}(z')$ : namely we conjecture that

$$\overline{T}(z) \tilde{Q}(z, \underline{s}) = (\phi_1(z, \underline{s}, w))^N \tilde{Q}(zq^2, \underline{s}^+) + (\phi_2(z, \underline{s}, w))^N \tilde{Q}(zq^{-2}, \underline{s}^-) \tag{5.11}$$

<sup>3</sup> The paper [6] primarily concerns the 8-vertex model, but the explicit formula (101) is obtained in the 6-vertex model limit.

and

$$[\overline{T}(z'), \tilde{Q}(z, \underline{s})] = 0$$

when  $n = 0$ . If Baxter's construction is valid, this should of course be true, and we have checked up to  $N = 10$  that the conjecture holds. Note that for the restriction  $s_1 = s_2 = 0$  and the choice (5.8) made here, we have

$$\phi_1(z, \underline{s}, w) = q \frac{w - z}{w - q^2 z} \quad \phi_2(z, \underline{s}, w) = 1.$$

When  $n \neq 0$ ,  $\delta(q^n)$  is well defined as a formal series in  $q$ , and  $\overline{Q}(z, \underline{s})$  obeys (4.17) and commutes with  $\overline{T}(z')$  by construction. As a consequence, it follows that (5.11) hold whenever  $q^n = 1$  (checks confirm this up to  $N = 8$ ). Checks also show that (5.11) are not valid for generic  $q$  and generic  $n \neq 0$ .

In summary: the situation for  $n = 0$  is that  $\overline{Q}(z, \underline{s})$  is ill defined but  $\tilde{Q}(z, \underline{s})$  appears to obey (5.11); for  $n \neq 0$ ,  $\overline{Q}(z, \underline{s})$  obeys (4.17) and commutes with  $\overline{T}(z')$  by construction, and  $\tilde{Q}(z, \underline{s})$  obeys (5.11) when  $q^n = 1$ .

Finally, we show in appendix B that

$$[\tilde{Q}(z, \underline{s}), \tilde{Q}(z', \underline{s}')] = 0 \tag{5.12}$$

for all  $n$ , including  $n = 0$ . This property implies in particular the commutativity between  $\overline{Q}(z, \underline{s})$  and  $\overline{Q}(z', \underline{s}')$  when  $n \neq 0$  (and when we restrict to  $s_1 = s_2 = 0$  as elsewhere in this section).

## 6. Discussion

To summarize: we have defined a  $U_q(b_+)$  representation  $M(z, \underline{s})$ , and used it to construct a  $U_q(b_-)$  valued operator  $Q(z, \underline{s})$  that obeys the generalized  $T$ - $Q$  relations (3.8) and commutes with the operator  $T(z')$ . In section 2.3, we have shown how, upon restricting  $\underline{s}$ ,  $Q(z, \underline{s})$  reduces to the operators  $Q_{\pm}(\lambda)$  constructed in terms of  $q$ -oscillator representations in [13].

We have then considered a representation of this object on the quantum space  $V_{w_1}^{(1)} \otimes \dots \otimes V_{w_N}^{(1)}$  corresponding to the 6-vertex model, and shown how the  $T$ - $Q$  relations are modified by the coefficients  $\phi_1$  and  $\phi_2$  appearing in (4.17).

We then proceeded to obtain the explicit form (5.9) for  $\overline{Q}(z, \underline{s})$ —valid for the  $s_1 = s_2 = 0$  case. When  $n \neq 0$ ,  $\overline{Q}(z, \underline{s})$  obeys relations (4.17), and commutes with  $\overline{T}(z')$  and  $\overline{Q}(z', \underline{s}')$ . When  $n = 0$ ,  $\overline{Q}(z, \underline{s})$  is ill defined, but  $\tilde{Q}(z, \underline{s})$  obeys (5.11) (up to  $N = 10$  at least) and commutes with  $\tilde{Q}(z', \underline{s}')$ . So in either case,  $n \neq 0$  or  $n = 0$ , our construction yields a well defined  $Q$ -matrix, either  $\overline{Q}(z, \underline{s})$  or  $\tilde{Q}(z, \underline{s})$ , that obeys the  $T$ - $Q$  relations and commutes both with  $\overline{T}(z')$  and with itself at different values of  $z$  and  $\underline{s}$ .

In section 5, we have also discussed the properties of  $\tilde{Q}(z, \underline{s})$  when  $n \neq 0$ , and there is a slightly mysterious aspect to this: while our algebraic construction of  $\overline{Q}(z, \underline{s})$  is valid for  $q$  generic, we find that  $\tilde{Q}(z, \underline{s})$  obeys relations (5.11) in the root of unity  $q^n = 1$  case. Clearly, this fact arises as a consequence of the delta function in (5.9), that in turn comes from the infinite-dimensional trace. Beyond this comment, we have no real understanding of this fact, but feel that it is still worth pointing it out, especially in the light of recent interest in the 6-vertex model at roots of unity [28].

We would like to mention some potential applications of our construction that we hope to consider in a future work. Firstly, there are intriguing similarities between our generalized  $T$ - $Q$  relations and the functional relations [21] linking the wavefunctions of the anharmonic Schrödinger equation in different Stokes sectors. The later functional relations also possess extra parameters over the conventional  $T$ - $Q$  relations that shift in a manner very similar to

ours (although there is one less parameter in the Stokes relations). Secondly, it is possible to derive Bethe equations from our generalized  $T$ - $Q$  relations that differ from the conventional equations in the possible  $\underline{s}$  dependence of the Bethe roots. It would be interesting to attempt to understand the solutions of such systems in the light of the fact that  $[Q(z, \underline{s}), Q(z', \underline{s}')] \neq 0$  in general. Finally, it would be interesting to consider representations of our  $Q(z, \underline{s})$  on a ‘continuous’ quantum space in order to be able to make connections with the constructions given in [12, 29].

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### Appendix A. Proof of proposition 3.1

Let  $\{a_1^{(1)}, a_2^{(1)}, \dots, a_N^{(1)}, a_1^{(2)}, a_2^{(2)}, \dots, a_M^{(2)}\}$  be a basis for  $A$ ,  $\{b_1, b_2, \dots, b_N\}$  be a basis of  $B$ , and  $\{c_1, c_2, \dots, c_M\}$  be a basis of  $C$ , such that the  $\mathcal{A}$  linear injection  $\iota$  and surjection  $\pi$  are given by

$$\begin{aligned} \iota : B &\rightarrow A & \pi : A &\rightarrow C \\ b_i &\mapsto a_i^{(1)} & a_i^{(1)} &\mapsto 0 & a_i^{(2)} &\mapsto c_i. \end{aligned}$$

The action of  $X$  on these basis vectors defines the following matrices:

$$\begin{aligned} Xa_j^{(1)} &= \sum_{i=1}^N {}^{(1)}X_{ij}^A a_i^{(1)} & Xa_j^{(2)} &= \sum_{i=1}^M {}^{(2)}X_{ij}^A a_i^{(2)} + \sum_{i=1}^N {}^{(3)}X_{ij}^A a_i^{(1)} \\ Xb_j &= \sum_{i=1}^N X_{ij}^B b_i & Xc_j &= \sum_{i=1}^M X_{ij}^C c_i. \end{aligned}$$

The required trace is given by

$$\mathrm{Tr}_A(X) = \sum_{i=1}^N {}^{(1)}X_{ii}^A + \sum_{i=1}^M {}^{(2)}X_{ii}^A.$$

Now note that we can write

$$\sum_{i=1}^N {}^{(1)}X_{ij}^A a_i^{(1)} = Xa_j^{(1)} = X\iota b_j = \iota Xb_j = \sum_{i=1}^N X_{ij}^B a_i^{(1)}$$

and

$$\sum_{i=1}^M X_{ij}^C c_i = Xc_j^{(1)} = X\pi a_j^{(2)} = \pi Xa_j^{(2)} = \sum_{i=1}^M {}^{(2)}X_{ij}^A c_i.$$

Hence, we have

$${}^{(1)}X_{ij}^A = X_{ij}^B \quad {}^{(2)}X_{ij}^A = X_{ij}^C,$$

and the proposition then follows.  $\square$



### Appendix B. Proof of $[\tilde{Q}(z, \underline{s}), \tilde{Q}(z', \underline{s}')] = 0$

In this section, we will prove that the operator  $\tilde{Q}(z, \underline{s})$ , defined in section 5, has the commutativity property

$$[\tilde{Q}(z, \underline{s}), \tilde{Q}(z', \underline{s}')] = 0. \quad (\text{B.1})$$

(Recall that we have restricted  $s_1 = s_2 = 0$  in the definition of  $\tilde{Q}(z, \underline{s})$ .)

From the definition of  $\tilde{Q}(z, \underline{s})$  given by (5.6), (5.9) and (5.10), it follows that

$$\begin{aligned} [\tilde{Q}(z, \underline{s}), \tilde{Q}(z', \underline{s}')]_{\underline{\alpha}}^{\underline{\gamma}} &= s_0^{-n} \sum_{\underline{\beta}} \left[ \exp\left(\frac{1}{2}iv\underline{\alpha} \cdot \underline{\beta} + \frac{1}{2}iv' \underline{\beta} \cdot \underline{\gamma}\right) \right. \\ &\quad \left. - \exp\left(\frac{1}{2}iv' \underline{\alpha} \cdot \underline{\beta} + \frac{1}{2}iv\underline{\beta} \cdot \underline{\gamma}\right) \right] \exp\left(\frac{1}{2}i\eta(\underline{\alpha} - \underline{\gamma}) \wedge \underline{\beta}\right) \end{aligned}$$

where we have introduced the notation

$$\underline{\alpha} \equiv (\alpha_1, \dots, \alpha_N) \text{ etc} \quad \underline{\alpha} \cdot \underline{\beta} = \sum_{j=1}^N \alpha_j \beta_j \quad \underline{\alpha} \wedge \underline{\beta} = \sum_{k=1}^N \sum_{j=1}^{k-1} (\alpha_j \beta_k - \alpha_k \beta_j). \quad (\text{B.2})$$

Now, it is simple to show that, given  $\underline{\alpha}$  belonging to the  $n$ th block (i.e.  $\sum_i \alpha_i = n$ ), every other element  $\underline{\gamma}$  belonging to the same block can be written as  $\underline{\gamma} = P\underline{\alpha}$ , where  $P$  is an element of the symmetric group  $S^N$  which has the following properties:

$$\begin{aligned} P &= P_{i_1, i_2} P_{i_3, i_4} \cdots P_{i_{k-1}, i_k} \quad i_1 < i_2, i_3 < i_4, \dots, i_{k-1} < i_k \\ [i_1, i_2] \supset [i_3, i_4] \supset \cdots \supset [i_{k-1}, i_k] \quad 1 \leq i_1, \dots, i_k \leq N \end{aligned} \quad (\text{B.3})$$

and  $P_{j,k}$  is the operator with action

$$P_{j,k}(\alpha_1, \dots, \alpha_j, \dots, \alpha_k, \dots, \alpha_N) = (\alpha_1, \dots, \alpha_k, \dots, \alpha_j, \dots, \alpha_N).$$

It follows from the definition of  $P$  that the following properties hold:

$$P^2 = 1 \quad P\underline{\alpha} \cdot P\underline{\beta} = \underline{\alpha} \cdot \underline{\beta}.$$

Using these properties we rewrite our commutator in the following form:

$$\begin{aligned} [\tilde{Q}(z, \underline{s}), \tilde{Q}(z', \underline{s}')]_{\underline{\alpha}}^{P\underline{\alpha}} &= \frac{s_0^{-n}}{2} \sum_{\underline{\beta}} \left[ \exp\left(\frac{1}{2}ivP\underline{\alpha} \cdot P\underline{\beta} + \frac{1}{2}iv' \underline{\alpha} \cdot P\underline{\beta}\right) \right. \\ &\quad \left. - \exp\left(\frac{1}{2}iv' P\underline{\alpha} \cdot P\underline{\beta} + \frac{1}{2}iv\underline{\alpha} \cdot P\underline{\beta}\right) \right] \left[ \exp\left(\frac{1}{2}i\eta(\underline{\alpha} - P\underline{\alpha}) \wedge \underline{\beta}\right) \right. \\ &\quad \left. - \exp\left(\frac{1}{2}i\eta(\underline{\alpha} - P\underline{\alpha}) \wedge P\underline{\beta}\right) \right]. \end{aligned}$$

We will prove by induction (over  $P$ ) the property

$$(\underline{\alpha} - P\underline{\alpha}) \wedge (\underline{\beta} - P\underline{\beta}) = 0 \quad \forall \underline{\alpha}, \underline{\beta}, P \quad (\text{B.4})$$

which implies that the previous commutator is zero.

Firstly, property (B.4) is easily shown when  $P = P_{j,k}$ ,  $1 \leq j < k \leq N$ . Next, let us make the inductive hypothesis that (B.4) is true for a given  $P$  of the form

$$\begin{aligned} P &= P_{i_1, i_2} P_{i_3, i_4} \cdots P_{i_{k-1}, i_k} \quad i_1 < i_2, i_3 < i_4, \dots, i_{k-1} < i_k \\ [i_1, i_2] \supset [i_3, i_4] \supset \cdots \supset [i_{k-1}, i_k] \quad 1 \leq i_1, \dots, i_k \leq N. \end{aligned} \quad (\text{B.5})$$

Now define  $P'$  by

$$P' = PP_j \quad \text{where } i < j \text{ and } [i_{k-1}, i_k] \supset [i, j]. \quad (\text{B.6})$$

Then we have that

$$(\underline{\alpha} - P'\underline{\alpha}) = (\underline{\alpha} - P\underline{\alpha}) + (\underline{\alpha} - P_{ij}\underline{\alpha})$$

and so (B.4) will be true for  $P'$  if

$$(\underline{\alpha} - P\underline{\alpha}) \wedge (\underline{\beta} - P_{ij}\underline{\beta}) + (\underline{\alpha} - P_{ij}\underline{\alpha}) \wedge (\underline{\beta} - P\underline{\beta}) = 0. \quad (\text{B.7})$$

Using the definition of the wedge product, the lhs is equal to

$$\begin{aligned} & \sum_{i < l < j} [\alpha_l - (P\alpha)_l](\beta_j - \beta_i) - \sum_{i < l \leq j} [\alpha_l - (P\alpha)_l](\beta_i - \beta_j) - \sum_{i \leq l < j} [\beta_l - (P\beta)_l](\alpha_j - \alpha_i) \\ & + \sum_{i < l \leq j} [\beta_l - (P\beta)_l](\alpha_i - \alpha_j). \end{aligned} \quad (\text{B.8})$$

However, it follows from the definitions (B.5) and (B.6) that we have

$$\alpha_l - (P\alpha)_l = 0 \quad \forall l \text{ such that } i \leq l \leq j$$

$$\beta_l - (P\beta)_l = 0 \quad \forall l \text{ such that } i \leq l \leq j$$

and therefore that expression (B.8) is equal to 0. It follows that (B.4) and hence (B.1) is true.  $\square$

## References

- [1] Lieb E H 1967 Residual entropy of square ice *Phys. Rev.* **162** 162–72
- [2] Lieb E H 1967 Exact solution of the  $F$  model of an antiferroelectric *Phys. Rev. Lett.* **18** 1046–8
- [3] Lieb E H 1967 Exact solution of the two-dimensional Slater KDP model of a ferroelectric *Phys. Rev. Lett.* **19** 108–10
- [4] Sutherland B 1967 Exact solution of a two-dimensional model for hydrogen-bonded crystals *Phys. Rev. Lett.* **19** 103–4
- [5] Baxter R J 1972 Partition function of the eight-vertex lattice model *Ann. Phys.* **70** 193–228
- [6] Baxter R J 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: I. Some fundamental eigenvectors *Ann. Phys.* **76** 1–24
- [7] Baxter R J 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: II. Equivalence to a generalized ice-type model *Ann. Phys.* **76** 25–47
- [8] Baxter R J 1973 Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain: III. Eigenvectors of the transfer matrix and Hamiltonian *Ann. Phys.* **76** 48–71
- [9] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
- [10] Takhtadzhan L A and Faddeev L D 1979 The quantum method of the inverse problem and the Heisenberg XYZ model *Russ. Math. Surv.* **34** 511–68
- [11] Pasquier V and Gaudin M 1992 The periodic Toda chain and a matrix generalization of the Bessel function recursion relations *J. Phys. A: Math. Gen.* **25** 5243–52
- [12] Bazhanov V V, Lukyanov S L and Zamolodchikov A B 1997 Integrable structure of conformal field theory: II.  $Q$ -operator and DDV equation *Commun. Math. Phys.* **190** 247–78
- [13] Antonov A and Feigin B 1997 Quantum group representations and the Baxter equation *Phys. Lett. B* **392** 115–22
- [14] Derkachov S E 1999 Baxter's  $Q$ -operator for the homogenous XXX spin chain *J. Phys. A: Math. Gen.* **28** 5299–316
- [15] Pronko G P 2000 On the Baxter's  $Q$ -operator for the XXX spin chain *Commun. Math. Phys.* **212** 687–701
- [16] Pronko G P 2000 On the Baxter's  $Q$ -operator for Toda chain *J. Phys. A: Math. Gen.* **33** 8251–66
- [17] Belavin A A, Odesskii A V and Usmanov R A 2001 New relations in the algebra of the Baxter  $Q$ -operators *Preprint hep-th/0110126*
- [18] Fioravanti D and Rossi M 2002 A braided Yang–Baxter algebra in a theory of two coupled lattice quantum KdV: algebraic properties and ABA representations *J. Phys. A: Math. Gen.* **35** 3647–82
- [19] Dorey P and Tateo R 1999 Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations *J. Phys. A: Math. Gen.* **38** L419–L425

- [20] Bazhanov V V, Lukyanov S L and Zamolodchikov Al B 2001 Spectral determinants for Schrödinger equation and  $Q$ -operators of conformal field theory *J. Stat. Phys.* **102** 567–76  
Bazhanov V V, Lukyanov S L and Zamolodchikov Al B 2000 *Proc. Baxter Revolution in Mathematical Physics (Canberra, 2000)*
- [21] Dorey P and Tateo R 1999 On the relation between Stokes multipliers and the  $T$ - $Q$  systems of conformal field theory *Nucl. Phys. B* **563** 573–602
- [22] Sklyanin E K 2000 Bäcklund transformations and Baxter's  $Q$ -operator *Integrable systems: From Classical to Quantum (CRM Proc. Lecture Notes vol 26)* (Providence, RI: American Mathematical Society) pp 227–50
- [23] Chari V and Pressley A 1991 Quantum affine algebras *Commun. Math. Phys.* **142** 261–83
- [24] Kuniba A, Nakanishi T and Suzuki J 1994 Functional relations in solvable lattice models: I. Functional relations and representation theory *Int. J. Mod. Phys. A* **9** 5215–66
- [25] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
- [26] Kulish P P 1991 Contraction of quantum algebras, and  $q$ -oscillators *Theor. Math. Phys.* **86** 108–10
- [27] Drinfeld V G 1986 *Proc. ICM* (Berkeley, CA: American Mathematical Society)
- [28] Fabricius K and McCoy B 2001 Completing Bethe's equations at roots of unity *J. Stat. Phys.* **104** 573–87
- [29] Pronko G and Sergeev S 2001 On the Baxter's  $Q$ -operator for Toda chain *J. Appl. Math.* **1** 47–68